MATH4210: Financial Mathematics Tutorial 5

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Convergence of r.v.s

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

$$\mathbb{P}[\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1$$

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

$$\lim_{n\to\infty} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

Convergence of r.v.s

Proposition

$X_n \to X$ *a.s. implies* $X_n \to X$ *in probability.*

Proposition

$X_n \to X$ *in probability implies there exists a subsequence of* X_n *converging to* X *a.s..*

$X_{n_k} \to X$ a.s. $k \to \infty$

Definition (Convergence in Law (in Distribution))

Let F_n and F be the c.d.f. of X_n and X for all $n \in \mathbb{N}$. $X_n \to X$ in Law (in Distribution) if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for any $x \in \mathbb{R}$ where F is continuous at x.

Proposition

$X_n \to X$ *in probability implies* $X_n \to X$ *in Law.*

(a) $\lim_{n\to\infty} E[|X_n \to X|^2] = 0 \Rightarrow \lim_{n\to\infty} E[|X_n - X|] = 0$

Recall Jensen inequality
if f is concave
$E[f(x)] \leq f(E(x))$

fix $n \in \mathbb{N}$

$E[|X_n - X|] = E[\sqrt{|X_n - X|^2}]$

$f(x) = \sqrt{x}$ concave

Definition

$\Rightarrow E[|X_n - X|] \leq \sqrt{E[|X_n - X|^2]} \Rightarrow 0$ as $n \to \infty$ $\square$

Given $p > 0$, denote by $X_n \to X$ in L^p if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$$

Question

(a). Show that $X_n \to X$ in L^2 implies $X_n \to X$ in L^1.
(b). Show that $X_n \to X$ in L^p implies $X_n \to X$ in probability.

(b) fix $\rho > 0$ fix $n \in \mathbb{N}$

markov inequality
$E[|X| > M] \leq \frac{1}{M} E[|X|]$

$P[\{\omega \in \Omega : |X_n - X| \geq \rho\}] = P[\{\omega \in \Omega : |X_n - X|^p \geq \rho^p\}]$

$\leq \frac{1}{\rho^p} E[|X_n - X|^p] \Rightarrow 0$ as $n \to \infty$

$\to 0$ as $n \to \infty$

$\square$

IBP $\int u\,dv = uv - \int v\,du$

$\partial_x u(t,x) = \int_{\mathbb{R}} f'(y+x)\, P_Z(y)\, dy$

IBP

$= f(y+x) P_Z(y) \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} P_Z'(y) f(y+x)\, dy$

(a) Define $Z \sim N(0, T-t)$ $\quad P_Z$: pdf of Z

$u(t,x) := E[f(Z+x)] = \int_{\mathbb{R}} f(y+x) P_Z(y)\, dy$

Question

Let $f : \mathbb{R} \to \mathbb{R}$ be such that for some constant $C > 0$, $f(x) < e^{C|x|}$ for all $x \in \mathbb{R}$. Define

$$u(t,x) = \mathbb{E}[f(B_T) | B_t = x] = \mathbb{E}[f(B_T - B_t + x)].$$

Show that

(a)

$$\partial_x u(t,x) = \mathbb{E}\left[\frac{B_T - B_t}{T - t} f(B_T - B_t + x)\right]$$

(b)

$$\partial_x^2 u(t,x) = \mathbb{E}\left[\frac{(B_T - B_t)^2 + (T - t)}{(T - t)^2} f(B_T - B_t + x)\right]$$

$P_Z'(y) = \frac{1}{\sqrt{2\pi(T-t)}} e^{\frac{-y^2}{2(T-t)}} \cdot \frac{-y}{(T-t)}$

$\partial_x u(t,x) = 0 + \int_{\mathbb{R}} \frac{1}{(T-t)} f(y+x) P_Z(y)\, dy = E\left[\frac{f(Z+x)}{T-t} \cdot Z\right] = E\left[\frac{f(B_T - B_t + x)}{T-t} \cdot B_T - B_t\right]$

$\sim \frac{(\mu, \sigma^2)}{P}$

$\frac{\partial}{\partial s} C_E(t, S_t) = N(d_1) + S_t \cdot \frac{\partial}{\partial s} N(d_1) - e^{-r(T-t)} K \frac{\partial}{\partial s} N(d_2)$

$\frac{\partial}{\partial s} N(d_1) = \frac{\partial}{\partial s} \int_{-\infty}^{d_1} f_Z(y)\,dy \quad Z \sim N(0,1) \quad f_Z(y)$: pdf of Z

$= f_Z(d_1) \frac{\partial}{\partial s} d_1$

Question

Consider the European call option price at time t:

$$C_E(t, S_t) = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

where $d_1 = \frac{ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = d_1 - \sigma\sqrt{T-t}$*. Compute*

(a) Delta: $\Delta = \partial_s C_E(t, S_t)$*.*

(b) Gamma: $\Gamma = \partial_s^2 C_E(t, S_t)$*.*

$\frac{\partial}{\partial s} d_1 = \frac{1}{\sigma\sqrt{T-t}} \frac{1}{S_t} = \frac{\partial}{\partial s} d_2$

$\frac{\partial}{\partial s} C_E(t, S_t) = N(d_1) + \frac{1}{\sigma\sqrt{T-t}} \left(\underbrace{S_t f_Z(d_1) - e^{-r(T-t)} K f(d_2)}_{=0} \right)$

$= N(d_1)$